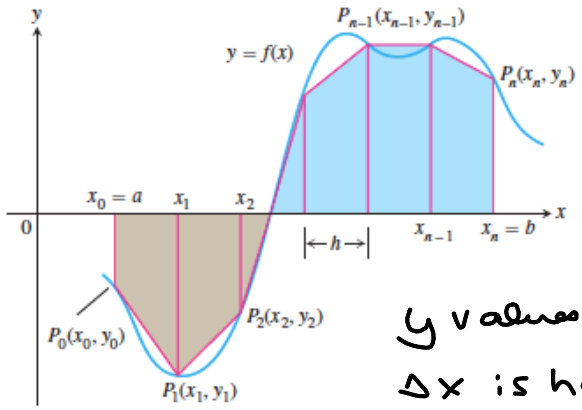
5.5 Trapezoidal Rule

Trapezoidal Approximations



y values are b_1 & b_2

Δx is height

Figure 5.31 The trapezoidal rule approximates short stretches of the curve $y = f(x)$ with line segments. To approximate the integral of f from a to b, we add the "signed" areas of the trapezoids made by joining the ends of the segments to the x-axis.

$$\int_a^b f(x)\,dx \approx h \cdot \frac{y_0 + y_1}{2} + h \cdot \frac{y_1 + y_2}{2} + \cdots + h \cdot \frac{y_{n-1} + y_n}{2}$$

$$= h\left(\frac{y_0}{2} + y_1 + y_2 + \cdots + y_{n-1} + \frac{y_n}{2}\right)$$

$$= \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right),$$

where

$$y_0 = f(a), \quad y_1 = f(x_1), \quad \cdots, \quad y_{n-1} = f(x_{n-1}), \quad y_n = f(b).$$

$$h = \frac{b-a}{n}$$

This is algebraically equivalent to finding the numerical average of LRAM and RRAM; indeed, that is how some texts define the Trapezoidal Rule.

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{h}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right),$$

where $[a, b]$ is partitioned into n subintervals of equal length $h = (b - a)/n$.

Equivalently,

$$T = \frac{\text{LRAM}_n + \text{RRAM}_n}{2},$$

where LRAM_n and RRAM_n are the Riemann sums using the left and right endpoints, respectively, for f for the partition.

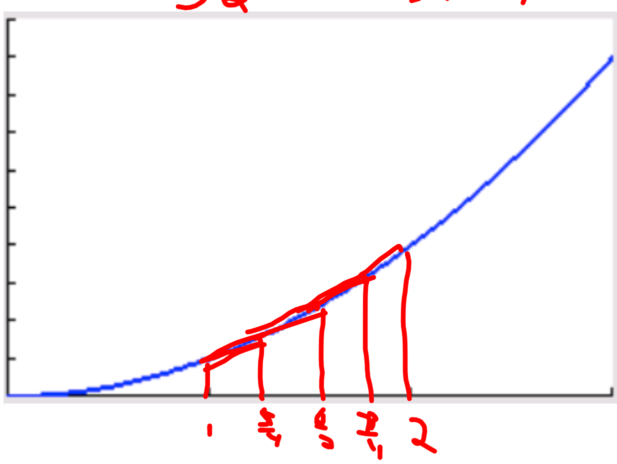
EXAMPLE 1 Applying the Trapezoidal Rule

Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the value of NINT $(x^2, x, 1, 2)$ and with the exact value.

$$\int_1^2 x^2\,dx \approx T$$

$$T = \frac{1}{4}\left(\frac{1}{2}\right)\left[\overset{y_0}{1^2} + 2\overset{y_1}{\left(\frac{5}{4}\right)} + 2\overset{y_2}{\left(\frac{6}{4}\right)} + 2\overset{y_3}{\left(\frac{7}{4}\right)} + 2^2\right]$$

$$= \frac{75}{32} = 2.34375$$



$$\int_1^2 x^2\,dx =$$

$$\frac{1}{3}x^3 \Big|_1^2$$

$$\frac{1}{3}(2)^3 - \frac{1}{3}(1)^3$$

exact

$$\frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

EXAMPLE 2 Averaging Temperatures

An observer measures the outside temperature every hour from noon until midnight, recording the temperatures in the following table.

x Time	N	1	2	3	4	5	6	7	8	9	10	11	M
y Temp	63	65	66	68	70	69	68	68	65	64	62	58	55

What was the average temperature for the 12-hour period?

$h = \Delta x = 1$

$$1\left(\frac{1}{2}\right)\left[63 + 2(65) + 2(66) + 2(68) + 2(70) + 2(69) + 2(68) + 2(68) + 2(65) + 2(64) + 2(62) + 2(58) + 55\right] = 782$$

avg value: $\frac{1}{b-a}\underbrace{\int_a^b f(x)\,dx}_{T}$

avg temp $= \frac{1}{12}(782) = 65.167°F$